Early Journal Content on JSTOR, Free to Anyone in the World

This article is one of nearly 500,000 scholarly works digitized and made freely available to everyone in the world by JSTOR.

Known as the Early Journal Content, this set of works include research articles, news, letters, and other writings published in more than 200 of the oldest leading academic journals. The works date from the mid-seventeenth to the early twentieth centuries.

We encourage people to read and share the Early Journal Content openly and to tell others that this resource exists. People may post this content online or redistribute in any way for non-commercial purposes.

Read more about Early Journal Content at http://about.jstor.org/participate-jstor/individuals/early-journal-content.

JSTOR is a digital library of academic journals, books, and primary source objects. JSTOR helps people discover, use, and build upon a wide range of content through a powerful research and teaching platform, and preserves this content for future generations. JSTOR is part of ITHAKA, a not-for-profit organization that also includes Ithaka S+R and Portico. For more information about JSTOR, please contact support@jstor.org.

(3, 4) In order that the third possibility be satisfied, a_2 and a_3 must both be equal to 0, and these are the only values of a_2 and a_3 which will also satisfy the fourth possibility.

Also solved by G. B. M. Zerr.

NUMBER THEORY AND DIOPHANTINE ANALYSIS.

120. **Proposed by L. E. DICKSON, Ph. D., The University of Chicago.**

Find the prime numbers p for which $x^2-pxz-px-z+p^2-3=0$ has more than two sets of positive integral solutions x, z, each $<p$.

Solution by H. S. VANDIVER, Bala, Pa.

About the time this problem was originally proposed, Professor Dickson wrote to me that he encountered it in determining those factors of $(p^2-1)^2$ which are of the form $1+px$, p prime. It will first be shown that the latter problem is equivalent to finding all the integers p, x, and z (p a prime) which satisfy

$$x^2-pxz-px-z+p^2-3=0\ldots(1).$$

Let $(p^2-1)^2=(1+px)(1+py)$, where x and y are positive integers. Expanding and dividing by p,

$$p^3-2p=x+y+pxy.$$

Hence there is necessarily an integer $z>0$, such that $x+y=pz$. By substitution and division by p,

$$z(px+1)=x^2+p^2-2.$$

Setting $z+1$ for z we obtain (1), and the two problems are equivalent.

Solving (1) for p,

$$2p=x(z+1)\pm\sqrt{[x^2(z+3)(z-1)+4(z+3)]}.$$

For this to hold, there is necessarily a positive integer α such that

$$x^2(z+3)(z-1)+4(z+3)=\alpha^2,$$

subject to the condition that $x(z+1)\pm\alpha$ be double a prime.

Putting $z+3=v$, $\alpha^2-vx^2(v-4)=4v$.

Let $v=k^2u$, where u is an integer not divisible by a square other than unity, then

$$u\beta^2-x^2(k^2u-4)=4\ldots(2),$$

where $\beta=\alpha/ku$, necessarily an integer.

The preceding analysis shows that (1) is soluble only for cases in which

(2) is soluble. It will suffice then to indicate a method leading to the complete solution of (2).

Regard u and k in (2) as fixed, and consider the relation as the representation of the integer 4 by a quadratic form in β and x, with coefficients u and k^2u-4. Consider the case where the determinant is of the form $k^2u^2-4u>4^2$. Then all the solutions of (2) can be obtained by using the theorem given in Serret's *Cours D'Algebre Supérieure* (Vol. I, p. 80). Suppose, first, that u and k are both odd; then the continued fraction to be considered is:

$$\sqrt{\frac{k^2u-4}{u}}=[k-1;\ 1,\ u'-1,\ 2,\ v'-1,\ 1,\ 2(ku-1),$$
$$1,\ v'-1,\ 2,\ u'-1,\ 1,\ 2(k-1);\ 1, \ldots],$$

where $ku=2u'+1$, $k=2v'+1$, $k>4$. (As a check, note that the period is reversible, $2(ku-1)$ being the middle term). If u is divisible by 2 but not by 4, and if k is odd, (2) reduces to the form

$$t\beta^2-x^2(k^2t-2)=2, \text{ where } u=2t,\ t \text{ not congruent to } 0 \pmod 2,$$

and the development is

$$\sqrt{\frac{k^2t-2}{t}}=[k-1;\ 1,\ kt-2,\ 1,\ 2(k-1);\ 1, \ldots] \quad (kt>2).$$

If $u=4s$, and k is odd or even, then (2) becomes

$$s\beta^2-x^2(k^2s-1)=1 \ldots (3),$$

and the fraction is

$$\sqrt{\frac{k^2s-1}{s}}=[k-1;\ 1,\ 2(ks-1),\ 1,\ 2(k-1);\ 1, \ldots] \quad (ks>1).$$

If k is even (2) reduces to the type (3), u arbitrary. For the few numerical values of k and u satisfying $k^2u^2-4u<4^2$, the Gaussian theory of quadratic forms may be used. The other exceptional cases of $k\leqq 4$, $kt\leqq 2$, $ks=1$, may be disposed of without difficulty. As an example, applying the Serret theorem to (2) for u and k odd, the development of $\sqrt{[(k^2u-4)/u]}$ shows that the third and eleventh complete quotients are the only ones in the period having the denominator 4. Put $k=5$, $u=3$, in (2), then using the convergent immediately preceding the eleventh quotient,

$$\beta/x=(\dot{4},\ 1,\ 6,\ 2,\ 1,\ 1,\ 28,\ 1,\ \dot{1},\ 2)=\tfrac{26275}{5401},$$

and we may verify that $3\times 26275^2-71\times 5401^2=4$.